

PROOF OF THE CONJECTURE OF KESKIN, SIAR AND KARAATLI

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ABSTRACT. In this paper among other results, we will prove the conjecture of Keskin, Siar and Karaatli on the Diophantine equation $x^2 - kxy + y^2 - 2^n = 0$.

1. INTRODUCTION

There have been much recent interest in the Diophantine equation

$$x^2 - kxy + y^2 + lx = 0 \quad (1)$$

for different values of the integers k and l . Marlewski and Zarzycki [4], considered equation (1) for $l = 1$, and proved that equation (1) has no positive solution for $l = 1$ and $k > 3$, but has an infinite number of solutions for $k = 3$ and $l = 1$. Keskin et al. in [2] and [3] considered equation (1) for $l = -1$ and proved that it has positive integer solutions for $k > 1$. Yuan and Hu [6] considered equation (1), with $l = 2$ or 4 and determined the values of the integer k for which equation (1) has an infinite number of positive solutions. Expanding on the work of Yuan and Hu [6], Keskin et al. in [2] and [3] considered equation (1) for $l = \pm 2^r$, with r a positive integer. They explained that in order to determine when equation (1) with $l = -2^r$, has an infinite number of positive integer solutions, one needs only to determine when the diophantine equation

$$x^2 - kxy + y^2 - 2^n = 0 \quad (2)$$

has an infinite number of positive integer solutions x and y for certain values of the non negative integer n . Similarly for $l = 2^r$ in equation (1), one needs only to consider the diophantine equation

$$x^2 - kxy + y^2 + 2^n = 0. \quad (3)$$

Keskin et al. solved equation (2) and equation (3) for $0 \leq n \leq 10$, and formulated the following conjecture.

Conjecture 1. (i) let n be an odd integer and $n > 2$. If $k > 2^n - 2$ then equation (2) has no positive integer solution. If $k \leq 2^n - 2$ and (2) has a solution, then k is even.
(ii) Let n be an even integer. If $k > 2^n - 2$, then equation (2) has no positive odd integer solution. If $k \leq 2^n - 2$ and equation (2) has a positive odd integer solution, then k is even.

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In this paper, among other results, we will prove Conjecture 1 in Theorem 3.1, and prove in Theorem 3.2 a result analogous to Conjecture 1.

2. PRELIMINARY RESULTS

In this section, we will recall some results that we will need for the proof of our theorems.

Let d be a positive integer which is not a perfect square, and consider the Pell equation

$$x^2 - dy^2 = 1. \quad (4)$$

It is well known that equation (4) always has a positive solution when $d \geq 2$. The least positive integer solution $x_1 + y_1\sqrt{d}$ to equation (4) is called the fundamental solution, and all positive integer solutions to (4) are given by

$$x_n + y_n\sqrt{d} = \left(x_1 + y_1\sqrt{d}\right)^n, \quad \text{with } n \geq 1.$$

Let C be a nonzero integer, and consider the Diophantine equation

$$u^2 - dv^2 = C. \quad (5)$$

Suppose that $u + v\sqrt{d}$ is a solution to equation (5). If $x + y\sqrt{d}$ is any solution of equation (4), then

$$\begin{aligned} u' + v'\sqrt{d} &= (u + v\sqrt{d})(x + y\sqrt{d}) \\ &= ux + vyd + (yu + vx)\sqrt{d} \end{aligned}$$

is also a solution of (5). The solution $u' + v'\sqrt{d}$ is said to be associated with the solution $u + v\sqrt{d}$. The set of all solutions associated with each other form a class of solutions of equation (5). Every class contains an infinity of solutions. We have the following lemmas.

Lemma 2.1. *If $u + v\sqrt{d}$ is the fundamental solution of a class K of the equation*

$$u^2 - dv^2 = N,$$

where N is a positive integer and if $x_1 + y_1\sqrt{d}$ is the fundamental solution of equation (4), then we have the inequalities

$$0 \leq v \leq \frac{y_1}{\sqrt{2}(x_1 + 1)}\sqrt{N},$$

and

$$0 < |u| \leq \sqrt{\frac{1}{2}(x_1 + 1)N}.$$

Lemma 2.2. *If $u + v\sqrt{d}$ is the fundamental solution of a class K of the equation $u^2 - dv^2 = -N$, where N is a positive integer and if $x_1 + y_1\sqrt{d}$ is the fundamental solution of equation (4), we have the inequalities*

$$0 < v \leq \frac{y_1}{\sqrt{2}(x_1 - 1)}\sqrt{N},$$

and

$$0 \leq |u| \leq \sqrt{\frac{1}{2}(x_1 - 1)N}.$$

For the proof of Lemma 2.1 and Lemma 2.2, see [5].

3. NEW RESULTS

In this section, we will prove Conjecture 1 in Theorem 3.1, and in Theorem 3.2 a result that is analogous to Conjecture 1 for the Diophantine equation

$$x^2 - kxy + y^2 = -2^n.$$

If $k = 0$, then equation (2) has finitely many solutions and equation (3) has no solution. We suppose in the sequel that $k \neq 0$.

Theorem 3.1. *Conjecture 1 is true*

Proof. (i) Let $n > 2$ be an odd integer. If (x, y) is a positive solution of equation (2), then clearly x and y have the same parity. If x and y are odd, then k is even. Suppose now that $2 \mid x$, then $2 \mid y$ and vice versa, and let $x = 2X$ and $y = 2Y$. Equation (2) yields

$$X^2 - kXY + Y^2 = 2^{n-2}. \quad (6)$$

Again, if X is even in (6), then Y is even and since $n-2$ is odd, we repeat the same process until all powers of 2 in x and y have been canceled. Therefore, we end up having the following equation

$$X^2 - kXY + Y^2 = 2^r,$$

where X , Y and r are positive odd integers. Hence k is clearly even. After a change of variables, equation (2) with n odd yields

$$u^2 - dv^2 = 2^n; \quad (7)$$

where $u = |x - \frac{k}{2}y|$, $y = v$ and $d = \frac{k^2}{4} - 1$.

Since k is even, then u and v are positive integers. If $k = 2$, equation (7) implies that $u^2 = 2^n$, which is impossible. Hence, $k > 2$, whereupon $d > 1$.

The solution $\frac{k}{2} + \sqrt{\frac{k^2}{4} - 1}$ is the fundamental solution to the Diophantine equation

$$x^2 - dy^2 = 1, \quad \text{where } d = \frac{k^2}{4} - 1.$$

If equation (2) has a positive solution with n an odd positive integer, then equation (7) has a positive solution. If $u + v\sqrt{d}$ is the fundamental solution of a class K of equation (7), then Lemma 2.1 implies that

$$0 \leq v \leq \frac{1}{\sqrt{2(\frac{k}{2} + 1)}} \sqrt{2^n}.$$

If $v = 0$, then equation (7) yields $u^2 = 2^n$, which is impossible. Therefore, $v \geq 1$ and the inequality above implies that $\sqrt{k+2} \leq \sqrt{2^n}$, i.e. $k \leq 2^n - 2$.

(ii) Let n be a positive even integer, and suppose that (x, y) is a solution to $x^2 - kxy + y^2 = 2^n$. If x and y are odd, then clearly k is even. Hence equation (2) yields $u^2 - dv^2 = 2^n$, where $u = |x - \frac{k}{2}y|$, $v = y$ and $d = \frac{k^2}{4} - 1$. Since $k \neq 0$, then $k \geq 2$, and d is a non negative integer. Lemma 2.1 implies that

$$0 \leq v \leq \frac{1}{\sqrt{2(\frac{k}{2} + 1)}} \sqrt{2^n}.$$

The solution $\frac{k}{2} + \sqrt{\frac{k^2}{4} - 1}$ is the fundamental solution to $x^2 - dy^2 = 1$, where $d = \frac{k^2}{4} - 1$. If $v = 0$, then equation (7) yields $u = 2^{n/2}$ and all solutions in the same classe as $(2^{n/2}, 0)$ are even. Hence, we suppose $v \geq 1$, and the last inequality implies that $\sqrt{2(\frac{k}{2} + 1)} \leq \sqrt{2^n}$, i.e. $k \leq 2^n - 2$. Therefore, if n is even and $k > 2^n - 2$, equation (2) has no positive odd solution and if $k \leq 2^n - 2$ and equation (2) has a positive odd integer solution, then k is even. \square

Theorem 3.2. (i) Let n be an odd integer and $n > 2$. If $k > 2^n + 2$, then the equation $x^2 - kxy + y^2 = 2^n$ has no positive integer solution. If $k \leq 2^n + 2$, and the equation $x^2 - kxy + y^2 = -2^n$ has a solution, then k is even.

(ii) Let n be an nonzero even integer. If $k > 2^n + 2$, then the equation $x^2 - kxy + y^2 = -2^n$ has no positive odd integer solution. If $k \leq 2^n + 2$ and the equation $x^2 - kxy + y^2 = -2^n$ has a positive odd integer solution, then k is even and 2 divide exactly k .

Proof. (i) let n be a positive odd integer and $n > 2$. Using the same reasoning as in the proof of Theorem 3.1, without loss of generality, we can suppose that the solutions x and y to (1) are odd. Hence k is even. Again, the same method in the proof of Theorem 3.1 and Lemma 2.2 imply that

$$1 \leq v \leq \frac{1}{\sqrt{2(\frac{k}{2} - 1)}} \sqrt{2^n}$$

whereupon, $\sqrt{k - 2} \leq \sqrt{2^n}$ i.e. $k \leq 2^n + 2$.

(ii) Suppose that n is even and that the equation $x^2 - kxy + y^2 = -2^n$ has a positive integer solution, then clearly k is even because $n \geq 1$, (the case $n = 0$ has been settled in [2]). Again the same method in the proof of Theorem 3.1 and Lemma 2.2 imply that $k \leq 2^n + 2$ and k even. If (x, y) is an odd solution to equation (1), then taking $x^2 - kxy + y^2 = 2^n$ modulo 4 implies that 2 divide exactly k . \square

Remark 3.1. It was proved in [3], that the diophantine equation $x^2 - kxy + y^2 = 2^n$ with $k = 2^n - 2$ has infinitely many solutions and in [2], that the Diophantine equation $x^2 - kxy + y^2 = -2^n$, with $k = 2^n + 2$ has infinitely many solutions. Hence the bounds of k in Theorem 3.1 and Theorem 3.2 are sharp.

Theorem 3.3. (i) Let $n > 2$ is an odd integer and p a prime such that $\left(\frac{2}{p}\right) = -1$. If equation (2) has a positive solution, then $\frac{k}{2} \not\equiv \pm 1 \pmod{p}$. In particular k is a multiple of 3.

(ii) Let $n > 2$ an odd integer, and p a prime such that $\left(\frac{2}{p}\right) = 1$. If equation (3) has a positive solution, then $\frac{k}{2} \not\equiv \pm 1 \pmod{p}$.

Proof. (i) If $n > 2$ is an odd integer and equation (2) has a positive solution, then the proof of Theorem 3.1 implies that k is even and the Diophantine equation $u^2 - \left(\frac{k^2}{4} - 1\right)v^2 = 2^n$ is solvable. Hence if p is an odd prime such that $\left(\frac{2}{p}\right) = -1$, then $\frac{k}{2} \not\equiv \pm 1 \pmod{p}$. By taking $p = 3$, we obtain that k is a multiple of 3.

(ii) The proof of (ii) is similar to (i) and will be omitted. \square

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